Arithmetic cusp shapes are dense

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Abstract

In this article we verify an orbifold version of a conjecture of Nimershiem from 1998. Namely, for every flat n–manifold M, we show that the set of similarity classes of flat metrics on M which occur as a cusp cross-section of a hyperbolic $(n+1)$–orbifold is dense in the space of similarity classes of flat metrics on M. The set used for density is precisely the set of those classes which arise in arithmetic orbifolds.

1 Main results

By a *flat n–manifold* we mean a closed manifold $M = \mathbf{R}^n/\Gamma$ where Γ is a discrete, torsion free, cocompact subgroup of $\mathrm{Isom}(\mathbf{R}^n)$. In analogy with Teichmüller theory, there is a contractible space $\mathcal{F}(M)$ of flat metrics on M coming from the standard flat structure on $\mathbf{R}^n$ and all the possible $\mathrm{Isom}(\mathbf{R}^n)$–conjugacy classes for Γ. We say that two flat metrics g_1, g_2 on M are *similar* if there exists an isometry between $(M, \alpha g_1)$ and (M, g_2) for some $\alpha \in \mathbf{R}^+$. We denote the equivalence class under similarity of a flat metric g by $[g]$ and the space of similarity classes of flat metrics on M by $\mathcal{S}(M)$.

An important relationship between flat and hyperbolic geometry is exhibited in the thick-thin decomposition of a hyperbolic manifold. Specifically, every finite volume, noncompact hyperbolic $(n+1)$–orbifold W has a thick-thin decomposition comprised of a compact manifold W_{core} with boundary components $M_1, \ldots, M_m$ and manifolds $E_1, \ldots, E_m$ of the form $M_j \times \mathbf{R}^{\geq 0}$. The manifolds E_j are called *cusp ends*, the manifolds M_j are called *cusp cross-sections*, and the union of W_{core} along the boundary with $E_1, \ldots, E_m$ recovers W topologically. The manifolds M_j are known to be flat n–manifolds and are totally geodesic boundary components of the manifold W_{core} equipped with the quotient metric coming from the path metric on its neutered space $\mathcal{N} \subset \mathbf{H}^{n+1}$. Indeed, a cusp cross-section M_j is naturally furnished with a flat metric g which is well-defined up to similarity and we call these similarity classes of flat metrics *realizable flat similarity classes* or *cusp shapes*. This article is devoted to the classification of the possible cusp shapes of a flat n–manifold occurring in the class of arithmetic $(n+1)$–orbifolds. The absence of a general geometric construction for hyperbolic orbifolds forces our restriction to orbifolds produced by arithmetic means.

Given this forced restriction, the picture we provide here is complete. Before stating our main results, we briefly survey some preexisting results and questions.

Motivated by the above picture and work of Gromov [3], Farrell and Zdravkovska [2] conjectured that every flat n–manifold arises as a cusp cross-section of a 1–cusped hyperbolic $(n+1)$–manifold and this is easily verified for $n = 2$. Indeed, the complement of a knot in S^3 is typically endowed with a finite volume, complete hyperbolic structure with one cusp (see [12]), and thus gives the realization of the 2–torus T^2 as a cusp cross-section of a 1–cusped hyperbolic 3–manifold. Likewise, the Klein bottle arises as a cusp cross-section of the 1–cusped Gieseking manifold (see [11]). However, Long and Reid [5] constructed counterexamples for $n = 3$ by showing that any flat 3–manifold arising as a cusp cross-section of a 1–cusped hyperbolic 4–manifold must have integral η–invariant; this works for all $n = 4k - 1$.

The failure of the conjecture of Farrell–Zdravkovska is far from total. Nimershiem [9] showed every flat 3–manifold arises as a cusp cross-section of a hyperbolic 4–manifold, and Long and Reid [6] proved every flat n–manifold arises as a cusp cross-section of an arithmetic hyperbolic $(n+1)$–orbifold. A more geometrically relevant question is precisely which cusp shapes occur on a given flat n–manifold. Via a counting argument, almost every similarity class on a flat n–manifold must fail to appear as a cusp shape. Despite this, Nimershiem [9] showed any for flat 3–manifold M the cusp shapes occurring in hyperbolic 4–manifolds are dense in the space of flat similarity classes on M, and conjectured [9, Conj. 2'] this for every flat n–manifold. Our main result is the verification of this conjecture in the orbifold category.

Theorem 1.1 (Cusp shape density). *For a flat n–manifold M, the set of cusp shapes of M occurring in hyperbolic $(n+1)$–orbifolds is dense in the space of flat similarity classes $\mathcal{S}(M)$.*

It is worth mentioning that the proof of Theorem 1.1 exhibits a dense subset of shapes of a uniform nature. These similarity classes are the image of **Q**–points of a **Q**–algebraic set under a projection map. From this one sees density occurs not as a function of small complexity in low dimensions but from the algebraic structure of these spaces. Moreover, these similarity classes of flat metrics are precisely those classes which occur in the cusp cross-sections of arithmetic hyperbolic $(n+1)$–orbifolds—see Theorem 3.7.

Using a modest refinement of Selberg's lemma, we verify the full conjecture for the n–torus.

Theorem 1.2. *For the n–torus $\mathbf{R}^n/\mathbf{Z}^n$, the set of cusp shapes of $\mathbf{R}^n/\mathbf{Z}^n$ occurring in hyperbolic $(n+1)$–manifolds is dense in the space of flat similarity classes $\mathcal{S}(\mathbf{R}^n/\mathbf{Z}^n)$.*

Acknowledgements

2 Preliminaries

2.1 Bieberbach groups and flat manifolds

We shall denote the affine, Euclidean, and similarity groups of $\mathbf{R}^n$ by $\mathrm{Aff}(n)$, $\mathrm{Euc}(n)$, and $\mathrm{Sim}(n)$. We call a discrete, torsion free subgroup Γ of $\mathrm{Euc}(n)$ a *Bieberbach group* when the quotient space $\mathbf{R}^n/\Gamma$ is a closed manifold and throughout the remainder of this article, Γ shall denote a Bieberbach group. We denote the space of faithful representations ρ of Γ in $\mathrm{Aff}(n)$ with Bieberbach images by $\mathcal{R}_f(\Gamma)$ (i.e., $\rho(\Gamma)$ is an $\mathrm{Aff}(n)$–conjugate of a Bieberbach group Γ'). Finally, $\mathcal{F}_f(\Gamma), \mathcal{S}_f(\Gamma)$ will denote the subspaces of the $\mathrm{Euc}(n)$ and $\mathrm{Sim}(n)$–character spaces consisting of those faithful characters whose images are Bieberbach.

Associated to each maximal compact subgroup K of $\mathrm{GL}(n;\mathbf{R})$ is the *orthogonal affine group* $\mathrm{O}_K(n) = \mathbf{R}^n \rtimes K$. As each K is conjugate in $\mathrm{GL}(n;\mathbf{R})$ to $\mathrm{O}(n)$, K is equal to $\mathrm{O}(B_K)$ for some symmetric, positive definite, bilinear form B_K on $\mathbf{R}^n$. When B_K is $\mathbf{Q}$–defined (i.e. $\mathbf{Q}$–valued on some $\mathbf{R}$–basis), $\mathrm{O}_K(n)$ is $\mathbf{Q}$–defined and we shall call subgroups commensurable with $\mathrm{O}_K(n;\mathbf{Z})$ $\mathbf{Q}$–*arithmetic subgroups* of $\mathrm{O}_K(n)$. We say ρ in $\mathcal{R}_f(\Gamma)$ is $\mathbf{Q}$–*arithmetic* if there exists a $\mathbf{Q}$–defined orthogonal affine group $\mathrm{O}_K(n)$ such that $\rho(\Gamma)$ is a $\mathbf{Q}$–arithmetic subgroup of $\mathrm{O}_K(n)$, and denote the subspace of $\mathcal{R}_f(\Gamma)$ of $\mathbf{Q}$–arithmetic representations by $\mathcal{R}_f(\Gamma;\mathbf{Q})$.

Recall that a manifold M is *flat* if M is diffeomorphic to $\mathbf{R}^n/\Gamma$ for some Bieberbach group Γ. The flat metric g induced by the standard inner product $\langle \cdot, \cdot \rangle$ on $\mathbf{R}^n$ supplies M with a flat metric called the *associated flat structure*. For future reference $\mathcal{F}(M)$ shall denote the space of all isometry classes of such metrics.

Theorem 2.1 ([13])**.** *The space of flat isometry classes on M is $\mathcal{F}_f(\pi_1(M))$. The space of flat similarity classes on M is $\mathcal{S}_f(\pi_1(M))$.*

These identifications are as real analytic spaces. Also, observe that any faithful representation ρ of $\pi_1(M)$ into $\mathrm{O}_K(n)$ whose image is Bieberbach endows M with a flat metric induced from the form B_K.

2.2 Hyperbolic space $\mathbf{H}^{n+1}$

The classical group $\mathrm{SO}(n+1,1)$ produces the symmetric space $\mathbf{H}^{n+1}$ known as hyperbolic $(n+1)$–space. For an explicit description, we equip $\mathbf{R}^{n+2}$ with a bilinear form B of signature $(n+1,1)$. *Hyperbolic $(n+1)$–space* is the $\mathbf{R}$–projectivization of the space of B–negative vectors (i.e., $v \in \mathbf{R}^{n+2}$ such that $B(v,v) < 0$) endowed with the Bergman metric associated to B. We denote hyperbolic $(n+1)$–space together with this metric by $\mathbf{H}^{n+1}$ and say $\mathbf{H}^{n+1}$ is *modelled on B* and call B a *model form*. The *boundary* of $\mathbf{H}^{n+1}$ in $\mathbf{RP}^{n+1}$ is the $\mathbf{R}$–projectivization of the space of B–null vectors (i.e., $v \in \mathbf{R}^{n+2}$ such that $B(v,v) = 0$) and we denote this set by $\partial\mathbf{H}^{n+1}$.

The isometry group of hyperbolic $(n+1)$–space $\mathbf{H}^{n+1}$ is afforded a *KAN* decomposition called *the Iwasawa decomposition*. The decomposition depends on the selection of a pair of boundary points $[v_0], [v_\infty]$. Most important for us here is the factor N which is

isomorphic to $\mathbf{R}^n$ (the factor K is a maximal compact subgroup and the factor A consists of all hyperbolic transformations fixing $[v_0]$ and $[v_\infty]$). Briefly, all such isomorphisms arise as follows. Let B be a model bilinear form for hyperbolic $(n+1)$–space and V_∞ be the B–orthogonal complement of a pair of linearly independent B–null vectors v_0 and v_∞ in $\mathbf{R}^{n+2}$. For any maximal compact K of $\mathrm{GL}(n;\mathbf{R})$ with associated positive definite form B_K, let

$$\psi \colon (\mathbf{R}^n, B_K) \longrightarrow (V_\infty, B_{|V_\infty})$$

be any isometric isomorphism. This induces the desired isomorphism $\eta \colon \mathbf{R}^n \longrightarrow N$ defined by

$$\eta(\xi) = \exp(\psi(\xi) v_\infty^* - v_\infty \psi(\xi)^*),$$

where $xy^*(\cdot) = B(\cdot, y)x$ is the outer pairing of x and y with respect to the form B. The injectivity and linearity of this map are straightforward to check. That this is surjective (i.e., N is isomorphic to $\mathbf{R}^n$) is less clear but well known and can be found in [11]. This extends to $\eta \colon \mathrm{O}_K(n) \longrightarrow \mathrm{Isom}(\mathbf{H}^{n+1})$, and produces

$$\eta(\mathrm{O}_K(n-1)) < \mathrm{Stab}([v_\infty]).$$

Viewing $\mathbf{H}^{n+1}$ as the coset space $\mathrm{Isom}(\mathbf{H}^{n+1})/K$, where K is a maximal compact subgroup, $\mathbf{H}^{n+1}$ is identified with $A \times N$ (the latter can be given a natural $\mathrm{Isom}(\mathbf{H}^{n+1})$–invariant metric for which this identification is an isometry). As $A = \mathbf{R}^+$, $\mathbf{H}^{n+1}$ supports the foliation

$$\bigcup_{t \in \mathbf{R}^+} \{t\} \times N$$

whose leaves are called *horospheres* and are said to be *centered at* $[v]$ if $[v] \in \partial \mathbf{H}^{n+1}$ plays the role of v_∞ in the discussion above on the isomorphism between N and $\mathbf{R}^n$.

By a *lattice* in $\mathrm{Isom}(\mathbf{H}^{n+1})$, we mean a discrete subgroup Λ such that $\mathbf{H}^{n+1}/\Lambda$ has finite volume. If $\mathbf{H}^{n+1}/\Lambda$ is compact, we say that Λ is *cocompact* and otherwise call Λ *noncocompact*. We call the quotient $\mathbf{H}^{n+1}/\Lambda$ a *hyperbolic $(n+1)$–orbifold* and when Λ is arithmetic, the quotient is referred to as an *arithmetic hyperbolic $(n+1)$–orbifold*. Of primary concern in this article are arithmetic lattices whose description is given by the following well known theorem—see for instance [4] or [8]. For the statement, recall that a pair of subgroups H_1, H_2 of a group G are *commensurable in the wide sense* if there exists $g \in G$ such that $(g^{-1}H_1g) \cap H_2$ is a finite index subgroup of both $g^{-1}H_1g$ and H_2 (when g is trivial, we say H_1 and H_2 are commensurable).

Theorem 2.2. *Every noncocompact arithmetic lattice in* $\mathrm{Isom}(\mathbf{H}^{n+1})$ *is commensurable in the wide sense with* $\mathrm{PO}_0(B;\mathbf{Z})$ *for some* $\mathbf{Q}$*–defined signature* $(n+1,1)$ *form B. Conversely, for $n \geq 3$, any such $\mathrm{PO}_0(B;\mathbf{Z})$ is a noncompact arithmetic lattice in* $\mathrm{Isom}(\mathbf{H}^{n+1})$.

For a lattice Λ in $\mathrm{Isom}(\mathbf{H}^{n+1})$ with associated orbifold $W = \mathbf{H}^{n+1}/\Lambda$, we say that W has a *cusp* at $[v]$ if $\Lambda \cap N \neq \{1\}$, where N is the group associated to $[v]$. In this case, the *maximal peripheral subgroup of Λ at* $[v]$ is the subgroup $\triangle_v(\Lambda) = \mathrm{Stab}([v]) \cap \Lambda$. By the Kazhdan–Margulis theorem, $\triangle_v(\Lambda)$ is virtually abelian and a maximal abelian

subgroup of $\triangle_v(\Lambda)$ is given by $L = \triangle_v(\Lambda) \cap N$. Moreover, the Kazhdan–Margulis theorem permits the selection of a horosphere $\{t\} \times N$ such that $\{t\} \times N/\triangle_v(\Lambda)$ is embedded in $\mathbf{H}^{n+1}/\Lambda$. In this case, we call $\{t\} \times N/\triangle_v(\Lambda)$ a *cusp cross-section* of the cusp at $[v]$.

Each cusp cross-section is equipped with a flat structure coming from the orthogonal affine group $O_K(n)$ of $\mathrm{Isom}(\mathbf{H}^{n+1})$ containing $\triangle_v(\Lambda)$. If B is a signature $(n+1,1)$ model form for $\mathbf{H}^n$, the associated form producing $O_K(n)$ is obtained by selecting a different boundary point $[w]$ and considering the restriction of B to the B–orthogonal complement of the $\mathbf{R}$–span of v and w. Up to similarity, this flat structure is independent of our selection of $[w]$ and thus yields a point in $\mathcal{S}(M)$. We say that $[g] \in \mathcal{S}(M)$ is *realizable* or is a *cusp shape* if there exists a hyperbolic $(n+1)$–orbifold W with cusp cross-section $(M', [g'])$ and a similarity transformation $f\colon (M, [g]) \longrightarrow (M', [g'])$.

3 Proofs of Theorem 1.1 and Theorem 1.2

3.1 Density of shapes

The proof of Theorem 1.1 is established in three steps. We first show that each representation ρ in $\mathcal{R}_f(\Gamma; \mathbf{Q})$ produces a similarity class that arises as a cusp shape in an arithmetic hyperbolic $(n+1)$–orbifold. After verifying the density of $\mathcal{R}_f(\Gamma; \mathbf{Q})$ in $\mathcal{R}_f(\Gamma)$, we construct continuous surjective maps from $\mathcal{R}_f(\Gamma)$ to both $\mathcal{F}_f(\Gamma)$ and $\mathcal{S}_f(\Gamma)$.

Proposition 3.1. *Let $\rho \in \mathcal{R}_f(\Gamma; \mathbf{Q})$ and $O_K(n)$ be any $\mathbf{Q}$–defined orthogonal affine group with $\rho(\Gamma)$ a $\mathbf{Q}$–arithmetic subgroup of $O_K(n)$. Then there exists an arithmetic lattice Λ in $\mathrm{Isom}(\mathbf{H}^{n+1})$ and an injection $\psi\colon O_K(n) \longrightarrow \mathrm{Isom}(\mathbf{H}^{n+1})$ such that $\psi(\rho(\Gamma))$ is a maximal peripheral subgroup of Λ.*

Note that by Mal'cev rigidity, ψ induces a similarity transformation between $\mathbf{R}^n/\Gamma$ and $\{t\} \times N/\psi(\rho(\Gamma))$. Consequently, in the sequel the transition from algebraic statements to geometric ones will be made without comment.

Proof. We commence the proof by noting that with the peripheral separability theorem in [7], we require only that $\psi(\rho(\Gamma))$ reside in Λ. To see this, assume that ψ is an injection of $\rho(\Gamma)$ into an arithmetic lattice Λ_0 and $\triangle$ is the maximal peripheral subgroup containing $\psi(\rho(\Gamma))$. As $\psi(\rho(\Gamma))$ is a finite index subgroup of $\triangle$, there exists a complete set of coset representatives $\gamma_1, \ldots, \gamma_r$ in $\triangle$ for $\triangle/\psi(\rho(\Gamma))$. For each γ_j, the peripheral separability theorem of [7] provides us with a finite index subgroup Λ_j of Λ_0 such that $\psi(\rho(\Gamma))$ is contained in Λ_j and $\gamma_j \notin \Lambda_j$. Setting

$$\Lambda = \bigcap_j \Lambda_j,$$

we obtain the needed pair ψ and Λ for the validity of the proposition.

It remains to find Λ_0 and ψ, a task achieved with the following line of reasoning. For the orthogonal affine group $O_K(n)$ with form B_K, define a model form B for $\mathbf{H}^{n+1}$ by

$$B = B_K \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and note that B is $\mathbf{Q}$–defined being a direct sum of such forms. We note that there is a $\mathbf{Q}$–defined injection of $O_K(n)$ into $PO_0(B)$ defined by sending $\mathbf{R}^n$ to the group N associated to the boundary point $[e_{n+1} + e_{n+2}]$ and K to the orthogonal group acting trivially on the 2–plane spanned by e_{n+1}, e_{n+2}. The former is given explicitly by

$$v \longmapsto \exp(v(e_{n+1} + e_{n+2})^* - (e_{n+1} + e_{n+2})v^*),$$

where we identify $\mathbf{R}^n$ inside $\mathbf{R}^{n+2}$ by taking the n–plane spanned by $e_1, \ldots, e_n$. The lattice Λ_0 in $PO_0(B)$ is obtained from an application of [10, Cor. 10.14]. Specifically, there exists Λ_0 in $PO_0(B)$ commensurable with $PO_0(B; \mathbf{Z})$ such that $\psi(\rho(\Gamma))$ is contained in Λ_0. $\qquad\qquad\square$

Proposition 3.2. *For every Bieberbach group Γ, $\mathcal{R}_f(\Gamma; \mathbf{Q})$ is dense in $\mathcal{R}_f(\Gamma)$.*

To prove Proposition 3.2, we need a pair of auxiliary lemmas.

Lemma 3.3. *If G is a topological group with dense subgroup H and X is a topological space with a continuous transitive G–action, then the H–orbit of any x in X is dense in X.*

Lemma 3.4. *For every crystallographic group Γ, $\mathcal{R}_f(\Gamma; \mathbf{Q})$ is nonempty.*

The first lemma is elementary and easily verified. The second lemma is a consequence of the Bieberbach theorems. Indeed, by the Bieberbach theorems, one is provided with an injection

$$\varphi \colon \Gamma \longrightarrow \mathbf{Z}^n \rtimes \mathrm{GL}(n; \mathbf{Z}).$$

One can then take the θ–average (see below for more on this) of the standard positive definite form on $\mathbf{R}^n$ to obtain a $\mathbf{Q}$–defined positive definite form B for which $\varphi(\Gamma)$ is an arithmetic subgroup of $O_{O(B)}(n)$.

Proof of Proposition 3.2. As expected, we seek to apply Lemma 3.3 and must ensure that the conditions are satisfied by $X = \mathcal{R}_f(\Gamma)$ and $G = \mathrm{Aff}(n)$. To begin, the topology on $\mathcal{R}_f(\Gamma)$ is the subspace topology induced by viewing $\mathcal{R}_f(\Gamma)$ as a subspace of the $\mathrm{Aff}(n)$–representation space. Visibly, the $\mathrm{Aff}(n)$–action on the representation space is continuous, and so by restriction the action of $\mathrm{Aff}(n)$ on $\mathcal{R}_f(\Gamma)$ is continuous. Less obvious is the transitivity of the $\mathrm{Aff}(n)$–action on $\mathcal{R}_f(\Gamma)$. However, this is precisely the statement of one part of the Bieberbach theorems. Thus, by Lemma 3.3, for $H = \mathbf{Q}^n \rtimes \mathrm{GL}(n; \mathbf{Q})$ and any ρ in $\mathcal{R}_f(\Gamma)$, the H–orbit of ρ is dense in $\mathcal{R}_f(\Gamma)$. We assert that for each α in H and $\rho \in \mathcal{R}_f(\Gamma; \mathbf{Q})$, the α–conjugate representation $\mu_\alpha \circ \rho$ is in $\mathcal{R}_f(\Gamma; \mathbf{Q})$, where

$$\mu_\alpha \colon \mathrm{Aff}(n) \longrightarrow \mathrm{Aff}(n)$$

is the inner automorphism given by conjugating by α. To see this, let $O_K(n)$ be a **Q**–defined orthogonal affine group for which $\rho(\Gamma)$ is a **Q**–arithmetic subgroup of $O_K(n)$. Conjugation by α yields an isomorphism

$$\mu_\alpha \colon O_K(n) \longrightarrow O_{\beta^{-1}K\beta}(n)$$

where $\beta \in GL(n;\mathbf{Q})$ is the linear factor (or second coordinate) for α. As β is an element of $GL(n;\mathbf{Q})$, the symmetric, positive definite form associated to $\beta^{-1}K\beta$ is **Q**–defined, being **Q**–equivalent to the **Q**–defined form B_K. Moreover, this isomorphism between $O_K(n)$ and $O_{\beta^{-1}K\beta}(n)$ is **Q**–defined. Therefore, any **Q**–arithmetic subgroup of $O_K(n)$ is mapped to a **Q**–arithmetic subgroup of $O_{\beta^{-1}K\beta}(n)$, and thus $\mu_\alpha \circ \rho(\Gamma)$ is a **Q**–arithmetic subgroup of a **Q**–defined orthogonal affine group as asserted. By Lemma 3.4, $\mathcal{R}_f(\Gamma;\mathbf{Q})$ is nonempty, and so there exists a $\mathcal{R}_f(\Gamma)$–dense H–orbit of representations in $\mathcal{R}_f(\Gamma;\mathbf{Q})$. $\qquad\Box$

Proof of Theorem 1.1. To begin, there exists a continuous surjective map

$$\mathcal{L} \colon \mathcal{R}_f(\Gamma) \longrightarrow \mathcal{F}_f(\Gamma)$$

given as follows. For ρ in $\mathcal{R}_f(\Gamma)$, as $\rho(\Gamma)$ is an $Aff(n)$–conjugate of a Bieberbach group, by the Bieberbach theorems $\rho(\Gamma)$ projects to a finite group θ in $GL(n;\mathbf{R})$. Taking the θ–average

$$B_\theta(x,y) = \frac{1}{|\theta|} \sum_{g \in \theta} \langle gx, gy \rangle$$

of the standard inner product $\langle \cdot, \cdot \rangle$ on $\mathbf{R}^n$ produces a maximal compact subgroup $K = O(B_\theta)$ such that $\rho(\Gamma)$ is contained in the orthogonal affine group $O_K(n)$. Up to post-composition with inner automorphisms of $Euc(n)$, there exists a unique S_ρ in $GL(n;\mathbf{R})$ conjugating $O_K(n)$ to $Euc(n)$. From this, we define $\mathcal{L}(\rho) = S_\rho^{-1}\rho S_\rho$.

If ρ_n is a sequence of representations in $\mathcal{R}_f(\Gamma)$ converging to ρ in $\mathcal{R}_f(\Gamma)$, the sequence $\mathcal{L}(\rho_n)$ converges to $\mathcal{L}(\rho)$ in $\mathcal{F}_f(\Gamma)$. For a free abelian group this is immediate since θ is trivial. For nontrivial θ, this follows from the convergence of the maximal compact subgroups K_n arising from the θ_n–average. We briefly explain this. For a sequence of representations ρ_n converging to ρ, let θ_n be the image of $\rho_n(\Gamma)$ under projection onto $GL(n;\mathbf{R})$. It follows that the groups θ_n converge to θ and thus B_{θ_n} converges to B_θ in the space of positive definite, symmetric matrices—we use the standard basis to associate these matrices to the θ–average forms. From this we see that the maximal compact subgroups K_n converge to K. The conjugating matrices S_{ρ_n} need not converge to the conjugating matrix S_ρ. However, up to left multiplication in $O(n)$, the sequence does converge and so the sequence of $S_{\rho_n}^{-1}\rho_n S_{\rho_n}$ converges to $S_\rho^{-1}\rho S_\rho$ in $\mathcal{F}_f(\Gamma)$. As this is the sequence $\mathcal{L}(\rho_n)$, we see that $\mathcal{L}(\rho_n)$ converges to $\mathcal{L}(\rho)$ in $\mathcal{F}_n(\Gamma)$.

The desired set of flat similarity classes is the image of $\mathcal{L}(\mathcal{R}_f(\Gamma;\mathbf{Q}))$ under the projection map

$$Pr \colon \mathcal{F}_f(\Gamma) \longrightarrow \mathcal{S}_f(\Gamma).$$

That this subset is dense is a consequence of the continuity and surjectivity of $\mathcal{L}$ in combination with Proposition 3.2. For the latter, if $\rho(\Gamma)$ resides in $Euc(n)$, the θ–average

of the standard form is the standard form and thus produces $O(n)$. In particular, we can take $S_\rho = I_n$ and hence $\mathcal{L}$ restricted to subset of $\mathrm{Euc}(n)$–representations is the standard projection onto $\mathcal{F}_f(\Gamma)$. It remains to show that each similarity class in $\mathrm{Pr}(\mathcal{L}(\mathcal{R}_f(\Gamma; \mathbf{Q})))$ does occur as a cusp shape of an arithmetic hyperbolic $(n+1)$–orbifold. By Proposition 3.1, for each $\rho \in \mathcal{R}(\Gamma; \mathbf{Q})$ with associated $\mathbf{Q}$–defined orthogonal affine group $O_K(n)$, there exists a faithful representation

$$\psi \colon O_K(n) \longrightarrow \mathrm{Isom}(\mathbf{H}^{n+1})$$

and an arithmetic lattice Λ such that $\psi(\rho(\Gamma))$ is a maximal peripheral subgroup of Λ. For the flat structure on $\mathbf{R}^n/\Gamma$ coming from $O_K(n)$ and the flat structure on the cusp cross-section associated to $\psi(\rho(\Gamma))$, this produces a similarity of this pair of flat manifolds. To obtain this for the associated class in $\mathrm{Pr}(\mathcal{L}(\rho))$, we argue as follows. It could be that the $\mathbf{Q}$–form B_K for K is not the θ–average of $\rho(\Gamma)$. If this is the case, simply replace K by $O(B_\theta)$, and notice that this too is a $\mathbf{Q}$–defined orthogonal affine group for which $\rho(\Gamma)$ is a $\mathbf{Q}$–arithmetic subgroup. Let M' be the associated flat manifold with this similarity class associated to ρ viewed as a representation into $O_{O(B_\theta)}(n)$. Making the same argument as before, we see that M' occurs as a cusp shape of an arithmetic hyperbolic $(n+1)$–orbifold. By construction, the flat manifold M'' with similarity class $\mathrm{Pr}(\mathcal{L}(\rho))$ is similar to M'. Hence, every class in the dense subset $\mathrm{Pr}(\mathcal{L}(\mathcal{R}_f(\Gamma; \mathbf{Q})))$ arises as a cusp shape of an arithmetic real hyperbolic $(n+1)$–orbifold. $\qquad\square$

Remark. Yves Benoist pointed out to us that by using only the subgroup separability result of [7], one can achieve density in a fixed commensurability class of arithmetic hyperbolic $(n+1)$–orbifolds for the n–torus. This argument can be extended to other flat manifolds given the work of Long–Reid [6].

For $X = \mathbf{C}$ or $\mathbb{H}$, cusp cross-sections of finite volume X–hyperbolic $(n+1)$–orbifolds are almost flat orbifolds modelled on the $(2n+1)$–dimensional Heisenberg group $\mathfrak{N}_{2n+1}$ or the $(4n+3)$–dimensional quaternionic Heisenberg group $\mathfrak{N}_{4n+3}(\mathbb{H})$. Generalizing the approach of Long and Reid, [7] gave the smooth classification of cusp cross-sections of X–hyperbolic n–orbifolds. Provided that an almost flat manifold arises topologically as a cusp cross-section, density follows with essentially the same argument.

Theorem 3.5. (a) *For an almost flat $(2n-1)$–manifold N modelled on $\mathfrak{N}_{2n-1}$, the space of realizable almost flat similarity classes in the cusp cross-sections of arithmetic complex hyperbolic n–orbifolds is either empty or dense in the space of almost flat similarity classes.*

(b) *For an almost flat $(4n-1)$–manifold N modelled on $\mathfrak{N}_{4n-1}(\mathbb{H})$, the space of realizable almost flat similarity classes in the cusp cross-sections of quaternionic hyperbolic n–orbifolds is either empty or dense in the space of almost flat similarity classes.*

As every Nil 3–manifold is diffeomorphic to a cusp cross-section of an arithmetic complex hyperbolic 2–orbifold (see [7]), Theorem 3.5 yields:

Corollary 3.6. *For a* Nil *3–manifold N, the space of* Nil *similarity classes that arise in the cusp cross-sections of arithmetic complex hyperbolic 2–orbifolds is dense in the space of* Nil *similarity classes.*

Finally, with the easily established converse of Proposition 3.1, we obtain our main theorem, the geometric classification of cusp cross-sections of arithmetic hyperbolic $(n+1)$–orbifolds.

Theorem 3.7 (Geometric classification theorem)**.** *For a flat n–manifold M, the set $Pr(\mathcal{L}(\mathcal{R}_f(\pi_1(M); \mathbf{Q})))$ is precisely the set of flat similarity classes on M that arise in cusp cross-sections of arithmetic hyperbolic $(n+1)$–orbifolds.*

This persists in the complex and quaternionic hyperbolic settings upon taking into account the above dichotomy.

3.2　Orbifold to manifold promotion

Theorem 1.2 is a consequence of the following proposition whose proof is essentially a reproduction of Borel's proof of Selberg's lemma [1, Prop. 2.2].

Proposition 3.8. *If $k/\mathbf{Q}$ is a finite extension and Λ a finitely generated subgroup of $\mathrm{GL}(n; k)$ with unipotent subgroup Γ, then there exists a torsion free, finite index subgroup Λ_0 of Λ such that Γ is contained in Λ_0.*

Proof. Let $\lambda_1, \ldots, \lambda_r$ be a finite generating set for Λ, $c_{i,j,\ell}$ be the (i, j)–coefficient of λ_ℓ, and R be the subring of k generated by $\{c_{i,j,\ell}\}$. By assumption, Γ is conjugate in $\mathrm{GL}(n; \mathbf{C})$ into the group of upper triangular matrices with ones along the diagonal. In particular, the characteristic polynomial $p_\gamma(t)$ for each γ in Γ is $(t-1)^n$. For any torsion element η in Λ, the characteristic polynomial $p_\eta(t)$ of η has only roots of unity for its zeroes. Since $k/\mathbf{Q}$ is a finite extension and n is fixed, there are only finitely many degree n monic polynomials in $k[t]$ having only roots of unity for their roots. Let $p_1(t), \ldots, p_s(t)$ denote those monic polynomials with coefficients in R with this property. As our concern is solely with nontrivial torsion elements, we further insist that each of the polynomials has a root distinct from 1. For each such polynomial $p_j(t)$, there are finitely many prime ideals $\mathfrak{p}$ of R such that $(t-1)^n = p_j(t)$ modulo $\mathfrak{p}$. To see this, we first exclude all prime ideals $\mathfrak{p}$ in R such that $\mathrm{char}(R/\mathfrak{p}) \leq n$. Since for each prime p of $\mathbf{Z}$ there are finitely many prime ideals $\mathfrak{p}$ of R such that $\mathrm{char}(R/\mathfrak{p}) = p$, this is a finite set. Next, as

$$p_j(t) - (t-1)^n = \sum_{m=1}^{n} \lambda_{m,j} t^m$$

is nonzero, there exists i such that $\lambda_{i,j}$ is nonzero. Since $\mathcal{O}_k$ is Dedekind, there are only finitely many prime ideals $\mathfrak{p}_{j,1}, \ldots, \mathfrak{p}_{j,\ell_j}$ such that $\lambda_{i,j} = 0 \mod \mathfrak{p}_{j,\ell_m}$ ($m = 1, \ldots, \ell_j$), and so for any other prime ideal $\mathfrak{q}$, it follows that $p_j(t) \neq (t-1)^n$ modulo $\mathfrak{q}$. Excluding this finite collection $\mathcal{P}_j$ of prime ideals of R, for any selection $\mathfrak{q} \notin \mathcal{P}_j$, we have $p_j(t)$ not equal to $(t-1)^n$ modulo $\mathfrak{q}$. Repeating this argument for each j, we obtain the desired ideal set $\mathcal{P}$. For $\mathfrak{q} \notin \mathcal{P}$, consider the reduction map $r_\mathfrak{q} \colon \mathrm{GL}(n; R) \longrightarrow \mathrm{GL}(n; R/\mathfrak{q})$. By our selection of $\mathfrak{q}$, no torsion element $r_\mathfrak{q}(\eta)$ can be contained in $r_\mathfrak{q}(\Gamma)$ as every

element of $r_{\mathfrak{q}}(\Gamma)$ has characteristic polynomial $(t-1)^n$ and η does not share this trait. Therefore, $r_{\mathfrak{q}}^{-1}(r_{\mathfrak{q}}(\Gamma))$ is a torsion free finite index subgroup of Λ containing Γ, as sought. $\qquad\square$

Remark. We note that by Weil's local rigidity theorem and the Zariski density of algebraic points, the assumption $\Lambda < \mathrm{GL}(n;k)$ for a finite extension $k/\mathbf{Q}$ is unnecessary.

Proof of Theorem 1.2. It suffices to show that for each ρ in $\mathcal{R}(\mathbf{Z}^n;\mathbf{Q})$ the induced representation given by Proposition 3.1 is such that $\mathbf{Z}^n$ is contained in a torsion free finite index subgroup of the target lattice Λ. By construction, the representation $\rho\colon \mathbf{Z}^n \longrightarrow \Lambda$ maps $\mathbf{Z}^n$ into a unipotent subgroup of Λ since the groups N in the Iwasawa decomposition are unipotent. As the target lattice Λ is arithmetic, Λ is finitely presentable ([10, Cor. 13.25]) and conjugate into the k–points of $\mathrm{Isom}(\mathbf{H}^{n+1})$ for some number field k. Thus Proposition 3.8 is applicable and yields a torsion free finite index subgroup Λ_0 of Λ such that $\mathbf{Z}^n$ is contained in Λ_0. Note that if $\mathbf{Z}^n$ is a maximal peripheral subgroup of Λ, $\mathbf{Z}^n$ is a maximal peripheral subgroup of Λ_0. In particular, we can realize the associated flat similarity class $[g]$ for ρ in a cusp cross-section of the associated arithmetic manifold for Λ_0. $\qquad\square$

For those infranil manifold groups realizable as lattices in their associated nilpotent Lie group, we say that the associated infranil manifold is a *niltorus*. For niltori modelled on either $\mathfrak{N}_{2n-1}$ or $\mathfrak{N}_{4n-1}(\mathbb{H})$, orbifold density is promoted to manifold density with an identical argument.

References

[1] A. Borel, *Compact Clifford-Klein forms of symmetric spaces*, Topology **2** (1963), 111–122.

[2] F. T. Farrell and S. Zdravkovska, *Do almost flat manifolds bound?*, Michigan Math. J. **30** (1983), no. 2, 199–208.

[3] M. Gromov, *Almost flat manifolds*, J. Differential Geom. **13** (1978), no. 2, 231–241.

[4] J.-S. Li and J. J. Millson, *On the first Betti number of a hyperbolic manifold with an arithmetic fundamental group*, Duke Math. J. **71** (1993), no. 2, 365–401.

[5] D. D. Long and A. W. Reid, *On the geometric boundaries of hyperbolic* 4–*manifolds*, Geom. Topol. **4** (2000), 171–178 (electronic).

[6] ———, *All flat manifolds are cusps of hyperbolic orbifolds*, Algebr. Geom. Topol. **2** (2002), 285–296 (electronic).

[7] D. B. McReynolds, *Peripheral separability and cusps of arithmetic hyperbolic orbifolds*, Algebr. Geom. Topol. **4** (2004), 721–755 (electronic).

[8] ———, *Cusps of arithmetic orbifolds*, Ph.D. thesis, University of Texas, 2006.

[9] B. E. Nimershiem, *All flat three-manifolds appear as cusps of hyperbolic four-manifolds*, Topology Appl. **90** (1998), no. 1-3, 109–133.

[10] M. S. Raghunathan, *Discrete subgroups of Lie groups*, Springer-Verlag, 1972.

[11] J. G. Ratcliffe, *Foundations of hyperbolic manifolds*, Springer-Verlag, 1994.

[12] W. P. Thurston, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. (N.S.) **6** (1982), no. 3, 357–381.

[13] _____, *Three-dimensional geometry and topology. Vol. 1*, Princeton University Press, 1997.

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